UDC 531.381

ALGORITHM FOR THE DERIVATION OF LIAPUNOV'S FUNCTION IN THE FORM OF A SUM OF SQUARES OF INTEGRALS

PMM Vol. 42, No. 5, 1978 pp. 946-949

A. N. VEISSENBERG
(Moscow)

(Received September 27, 1977)

Conditions of motion stability of a system admitting first integrals are obtained in the form of sufficient conditions of zero solution uniqueness of a nonlinear system. The problem of stability of permanent rotations of a heavy solid body with a single fixed point is illustrated here by the establishment of three sufficient conditions of such stability. Two of these coincide with those derived earlier [1], while the third is more general. The proposed procedure for the derivation of Liapunov's function from integrals of motion is a synthesis of a number of known methods [2 —4]. The problem used here as an illustration was considered by several authors (see, e. g., [5 —9]). A new set of permanent rotations is formulated directly on the admissible arc on the Staude cone in conformity with Rumiantsev's theories.

1. Let $U_i(x) = \text{const}$ $(i = 1, 2, \ldots, k)$ be the first integrals of motion of some mechanical system (x is an element of an n-dimensional space). Functions $U_i(x)$ are assumed determinate in the neighborhood of zero and continuous there together with their first order derivatives. If $U_i(0) = 0$ and the identity $x \equiv 0$ define the unperturbed motion of the input mechanical system, then

$$V(x) = \sum_{i=1}^{k} U_i^2(x)$$

may be taken as the Liapunov function for investigating such system stability.

Function $V(x)$ is positive definite if and only if there exists a cube $S(0, a) = \{x: |x^j| < a,\ j = 1, \ldots, n;\ a > 0\}$ in which the zero solution of the system of equations

$$U_1(x) = 0, \ldots, U_k(x) = 0 \tag{1.1}$$

is unique.

Let us assume that the integrals are holomorphic ($\alpha_1^i, \ldots, \alpha_n^i, \alpha_{sj}^i$ are real constants), i. e.

$$U_i(x) = \alpha_1^i x^1 + \ldots + \alpha_n^i x^n + \sum_{s,j=1}^{n} \alpha_{sj}^i x^s x^j + o(\|x\|^2) \tag{1.2}$$

that the rank of matrix $\|\alpha^i{}_j\|$ is $(k-1)$ and the nonzero determinant of the $(k-1)$-st order is located in the upper left-hand corner. We assume that the diagonal minors of that determinant are nonzero. On these assumptions it is possible to find a cube $S(0, a)$ such that the set of points $(x^1, \ldots, x^r, x^{r+1}, \ldots x^n)$ contained in it satisfies the first $r \leqslant k-1$ equations of system (1.1) and is defined by the continuously differentiable functions [10]

$$x^i = \psi_r{}^i (x^{r+1}, \ldots, x^n), \quad \psi_r{}^i (0, \ldots, 0) = 0, \quad i = 1, \ldots, r \tag{1.3}$$

The uniqueness of solution $\psi_r{}^i$ implies that when a certain nonzero vector satisfies these equations, at least one of its components $x_0^{r+1}, \ldots, x_0{}^n$ is nonzero.

We shall indicate k sufficient conditions of stability of the zero solution of the system of differential equations which admits integrals (1.2). These consitions successively extend in the sense that when the i-th condition is satisfied, then all conditions numbered $i+1$ to k are automatically satisfied.

T h e o r e m. There exist k successively extending sufficient conditions of stability of the zero solution of the system of differential equations that admit integrals (1.2). These conditions are determined by the sufficient conditions for the functional determinant

$$\Delta (x) = | a_{ij} |, \; a_{ij} = \alpha_j{}^i, \; j = 1, \ldots, k - 1$$
$$a_{ik} = U_i (x); \; i = 1, \ldots k \tag{1.4}$$

or any of the expressions

$$f_{r+1} = \Delta [\psi_r{}^1 (x^{r+1}, \ldots, x^n), \ldots, \psi_r{}^r (x^{r+1}, \ldots, x^n), x^{r+1}, \ldots, x^n] \tag{1.5}$$
$$r = 1, \ldots, (k-1)$$

to have the property of fixed sign.

P r o o f. We represent the system of Eqs.(1.1) in the form

$$\alpha_1{}^i x^1 + \ldots + \alpha_n{}^i x^n = - U_i (x) + \sum_{s=1}^n \alpha_s{}^i x^s, \quad i = 1, \ldots, k \tag{1.6}$$

and shall show that when any of the k conditions of the theorem are satisfied, there exists a number $a > 0$ such that in the cube $S(0, a)$ the solution $x = 0$ of the system of Eqs. (1.1) is unique. Let us compare the rank of matrix $\| \alpha_j{}^i \|$ $(i = 1, \ldots, k; \; j = 1, \ldots, n)$ with that of a matrix whose last column elements b^i are of the form

$$b^i = - U_i (x) + \sum_{s=1}^n \alpha_s{}^i x^s$$

and consider the k-th order minor $\Delta_k (x)$ of the latter which is composed of $(k-1)$ -st firsts and the last columns. Obviously $\Delta_k (x) = - \Delta (x)$. If function $\Delta (x)$ is of fixed sign, there exists a cube $S(0, a)$ $(a > 0)$ in which $\Delta (x) > 0$ and the solution $x = 0$ of the system of Eqs. (1.1) is unique when $x \neq 0$. To prove this we assume the existence of a nonzero solution $x_0 \in S(0, a)$ of that system, and consider the ancillary linear system of equations obtained by the substitution of x_0 into its right-hand side. Since $\Delta_k (x_0) \neq 0$, hence by the Kronecker —Capelli theorem the ancillary linear system has no solution and, consequently, vector x_0 cannot be a solution of the system of Eqs. (1.1).

Let us now assume that function f_{r+1} is of fixed sign. Then there exists an $(n - r)$ -dimensional cube $|x^{r+1}| < a, \ldots, |x^n| < a \; (a > 0)$ in which that function is nonzero, except at points $x^{r+1} = \ldots = x^n = 0$. The solution $x = 0$ of the system of Eqs. (1.1) in the n -dimensional cube $S(0, a)$ is unique. To prove this we assume that $x_0 \neq 0$, $x_0 \in S(0, a)$ is the solution of system (1.1). Then at least one of the components $x_0^k, \ldots, x_0^n$ must be nonzero and

$$\Delta(x_0) = \Delta[\psi_r^1(x_0^{r+1}, \ldots, x_0^n), \ldots, \psi_r^r(x_0^{r+1}, \ldots, x_0^n), x_0^{r+1}, \ldots, x_0^n]$$

Since in the ancillary linear system of equations introduced above $\Delta_k(x_0) \neq 0$, hence x_0 cannot be a solution of Eqs. (1.6) and (1.1).

Let us show that the stability conditions extend successively. The constant sign property of $\Delta(x)$ implies that function f_2 has the same property. Let function f_{r+1} be of fixed sign and at least one the quantities $x^{r+2}, \ldots, x^n$ be nonzero. Then function

$$f_{r+1}[\psi_{r+1}^{r+1}(x^{r+2}, \ldots, x^n), x^{r+2}, \ldots, x^n] \tag{1.7}$$

is positive. Taking into account the identity

$$\psi_{r+1}^i(x^{r+2}, \ldots, x^n) \equiv \psi_r^i[\psi_{r+1}^{r+1}(x^{r+2}, \ldots, x^n), x^{r+2}, \ldots, x^n], \qquad i = 1, \ldots, r \tag{1.8}$$

we find that function (1.7) satisfies the identity

$$\Delta[\psi_{r+1}^1(x^{r+2}, \ldots, x^n), \ldots, \psi_{r+1}^r(x^{r+2}, \ldots, x^n), \psi_{r+1}^{r+1}(x^{r+2}, \ldots, x^n), x^{r+2}, \ldots, x^n] = f_{r+2}$$

i.e. f_{r+2} is of fixed sign, since it is a function of arguments $x^{r+2}, \ldots, x^n$. The theorem is proved.

The principle formulated above goes back to Volterra [2] who had shown that in the case of three-dimensional vector x and two integrals $U_1(x)$ and $U_2(x)$ all isolated points of the system of equations $U_1(x) = 0$ and $U_2(x) = 0$ represent stable equilibrium positions. Function $\Delta(x)$ is a linear bundle of first integrals in which there are no linear terms. The sufficient condition for that bundle to be of fixed sign determine the first stability condition which, although the most restricted, is the most convenient for the analysis. The most extended is the k-th condition; it coincides with the stability condition that follows from the Routh theorem with Liapunov's supplement [11, 12]. Note that Eqs. (1.1) define an integral manifold similar to that encountered in solutions of the inverse problems of dynamics [13].

2. Let us apply the procedure set forth in Sect. 1 for determining the stability region of rotation of a heavy solid body with a single fixed point in direct formulation on the Staude cone. The equations of motion are of the form

$$dx / dt = Px + \chi(x) \tag{2.1}$$

$$P = \begin{Vmatrix} \omega F & -F \\ Q & H \end{Vmatrix}, \quad F = \begin{Vmatrix} 0 & \gamma & -\beta \\ -\gamma & 0 & \alpha \\ \beta & -\alpha & 0 \end{Vmatrix}$$

$$Q = G \begin{Vmatrix} 0 & z_0 A^{-1} & -y_0 A^{-1} \\ -z_0 B^{-1} & 0 & x_0 B^{-1} \\ y_0 C^{-1} & -x_0 C^{-1} & 0 \end{Vmatrix}$$

$$H = \omega \begin{Vmatrix} 0 & (B-C)\gamma\, A^{-1} & (B-C)\beta\, A^{-1} \\ (C-A)\gamma\, B^{-1} & 0 & (C-A)\alpha\, B^{-1} \\ (A-B)\beta\, C^{-1} & (A-B)\alpha\, C^{-1} & 0 \end{Vmatrix}$$

$$\chi^1(x) = x^2x^6 - x^5x^3, \quad \chi^2(x) = x^3x^4 - x^6x^1$$
$$\chi^3(x) = x^1x^5 - x^4x^2, \quad \chi^4(x) = (B - C)A^{-1}x^5x^6$$
$$\chi^5(x) = (C - A)B^{-1}x^4x^6, \quad \chi^6(x) = (A - B)C^{-1}x^4x^5$$

where $A > B > C > 0$ are moments of inertia, $x_0 > 0, y_0 > 0, z_0 > 0$ are the coordinates of the center of gravity, and α, β, γ, and ω determine the specified permanent rotation. Equation (2.1) admits the integrals

$$\begin{aligned} U_1(x) &= 2\alpha x^1 + 2\beta x^2 + 2\gamma x^3 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 0 \\ U_2(x) &= A\alpha\omega x^1 + B\beta\omega x^2 + C\gamma\omega x^3 + A\alpha x^4 + B\beta x^5 + \\ &\quad C\gamma x^6 + Ax^1x^4 + Bx^2x^5 + Cx^3x^6 = \text{const} \\ U_3(x) &= 2Gx_0x^1 + 2Gy_0x^2 + 2Gz_0x^3 + 2A\alpha\omega x^4 + \\ &\quad 2B\beta\omega x^5 + 2C\gamma\omega x^6 + A(x^4)^2 + B(x^5)^2 + C(x^6)^2 = \text{const} \end{aligned} \tag{2.2}$$

where U_1, U_2, and U_3 represent the trivial first integral, the integral of areas, and the integral of energy, respectively. From (1.4) and (2.2) we have

$$\Delta(x) = 2\alpha\beta\omega(B - A)\Delta^\circ(x^1, x^2, x^3, x^4, x^5, x^6)$$
$$\Delta^\circ(x^1, x^2, x^3, x^4, x^5, x^6) = -G\Delta_1^\circ + \Delta_2^\circ$$
$$\Delta_1^\circ = \alpha^{-1}x_0(x^1)^2 + \beta^{-1}y_0(x^2)^2 + \gamma^{-1}z_0(x^3)^2$$
$$\Delta_2^\circ = A(\omega x^1 - x^4)^2 + B(\omega x^2 - x^5)^2 + C(\omega x^3 - x^6)^2$$

The first two stability conditions are determined by the sufficient conditions of the fixed sign property of functions $\Delta^\circ(x^1, x^2, x^3, x^4, x^5, x^6)$ and $\Delta^\circ[\psi_1^1(x^2, x^3), x^2, x^3, x^4, x^5, x^6]$, where $\psi_1^1(x^2, x^3)$ is the solution of equation $U_1(x) = 0$ for x^1. The second of these conditions coincides with conditions in [1] and is defined by the inequalities

$$\begin{aligned} &-(\alpha^{-1}\beta^2x_0 + \alpha^2\beta^{-1}y_0) > 0 \\ &(\alpha^{-1}\beta^2x_0 + \alpha^2\beta^{-1}y_0)\gamma^{-1}z_0 + \alpha^{-1}\beta^{-1}\gamma^2x_0y_0 > 0 \end{aligned} \tag{2.3}$$

The third condition is determined by the sufficient conditions of fixed sign property of function

$$\Delta^\circ[\psi_2^1(x^3, x^4, x^5, x^6), \psi_2^2(x^3, x^4, x^5, x^6), x^3, x^4, x^5, x^6] \tag{2.4}$$

where ψ_2^1 and ψ_2^2 are obtained by solving the system of equations $U_1(x) = 0$ and $U_2(x) = 0$ for x^1 and x^2. Using the Lagrange method for reducing a quadratic to a sum of squares it is possible to show that the third stability condition shows the existence on the arc $(x, -y)$ of the Staude cone of a set of permanent rotations that is wider that in [1].

Note that stability is only possible in the critical case of two pairs of pure imaginary roots and a double zero root [14]. Nonmultiple elementary divisors of matrix P correspond to a zero multiple root, since two linearly independent solutions can be indicated for the equation $Pu = 0$.

R e m a r k. Function $\Delta(x)$ is the quadratic integral for Eq. (2.1) which is unique and correct within the constant multiplier. We prove this using the equation

$$\sum_{i=1}^{6}[p_{i1}x^1 + \ldots + p_{i6}x^6 + \chi^i(x)]\frac{\partial V}{\partial x^i} = 0$$

whose solution is sought in the form $V = x'Xx$, where $X = \| x_{ij} \|_1^6$ is a symmetric matrix with indeterminate coefficients. The latter are determined by using equations $x'X\chi(x) = 0$ and $x'(P'X + XP)x = 0$ which reduce to a system of algebraic equations in x_{ij} which has a unique solution.

REFERENCES

1. Rumiantsev, V. V., Stability of permanent rotations of a heavy solid body with a single fixed point. PMM, Vol. 20, No. 1, 1956.
2. Volterra, V., Sur la théorie des variations des latitudes. Acta Math., Vol., 22, 1899.
3. Pozharitskii, G. K., On the derivation of Liapunov's functions from the integrals of equations of perturbed motion. PMM, Vol. 22, No. 2, 1958.
4. Rubanovskii, V. N. and Stepanov, S. Ia., On the Routh theorem and Chetaev's method of derivation of Liapunov's function from integrals of equations of motion, PMM, Vol. 33, No. 5, 1969.
5. Kuz'min, P. A., Steady motions of solid body and their stability in a central gravitational field. Tr. Internat. Conf. on Appl. Theory of Motion Stability and in Analyt. Mechanics. Kazan', 1964.
6. Rubanovskii, V. N., On bifurcation and stability of steady motions of systems with known first integrals. Collection: Problems of Analysis of Motion Stability and Stabilization, No. 1, Moscow, VTs Akad. Nauk SSSR, 1975.
7. Tatarinov, Ia. V., Forms of classic integrals of the problem of solid body rotation about a fixed point. Vestn. MGU, Der. Matem. Mekh., No. 6, 1974.
8. Sergeev, V. S., On the stability of permanent rotation of a heavy solid body about a fixed point. PMM, Vol. 40, No. 3, 1976.
9. Burlakova, L. A. and Irtegov, V. D., Singularities of Lagrange transformation and steady motions. All-Union Conf. on the Qualitative Theory of Differential Equations and Methods of Teaching the Theory of Differential Equations in the Pedagogical Institute, Riazan', 1976.
10. Nikol'skii, S. M., Course of Mathematical Analysis, Vol. 1—2. Moscow, "Nauka", 1975.
11. Liapunov, A. M., Collected Works, Vol. 1 Moscow—Leningrad, Izd. Akad. Nauk SSSR, 1954.
12. Rumiantsev, V. V., On the stability of steady motions of satellites. Moscow, VTs Akad. Nauk SSSR, 1967.
13. Galiullin, A. S., Derivation of equations of motion. Differential Equations Vol. 13, No. 2, 1977.

14. V e i s s e n b e r g, A. N., Investigation of permanent rotation stability of a heavy solid body with a single fixed point. All-Union Conf. on the Qualitative Theory of Differential Equations and Methods of Teaching the Theory of Differential Equations in the Pedagogical Inst., Riazan', 1976.

Translated by J.J. D.